# AN EPIDEMIOLOGICAL APPROACH
# TO THE SPREAD OF POLITICAL THIRD PARTIES

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Abstract. Third political parties are influential in shaping American politics. In this work we study the spread of a third party ideology in a voting population where we assume that party members/activists are more influential in recruiting new third party voters than non-member third party voters. The study uses an epidemiological metaphor to develop a theoretical model with nonlinear ordinary differential equations as applied to a case study, the Green Party. Considering long-term behavior, we identify three threshold parameters in our model that describe the different possible scenarios for the political party and its spread. We also apply the model to the study of the Green Party's growth using voting and registration data in six states and the District of Columbia to identify and explain trends over the past decade. Our system produces a backward bifurcation that helps identify conditions under which a sufficiently dedicated activist core can enable a third party to thrive, under conditions which would not normally allow it to arise. Our results explain the critical role activists play in sustaining grassroots movements under adverse conditions.

1. **Introduction.** The 2000 United States presidential election was for many a testimony to the impact of third parties in a traditionally bipartisan government. Ralph Nader, the presidential candidate for the Green Party, won 2% of the popular vote, a percentage that many attribute to the defeat of Democratic candidate Al Gore [35]. The Green Party captured a seemingly insignificant number of votes

relative to majority percentages, yet its presence in the election ultimately served to shape American politics for the years following. This incident demonstrates how third parties, often emerging as grassroots movements (i.e., movements at the local level rather than at the center of major political activity), can ultimately impact at the national level, hence prompting the need to study their emergence and spread within a voting population.

Third parties are defined as political parties operating along with two major parties in a bipartisan system over a limited period of time (where we define a limited period of time as a range of a few years). For the purposes of this paper we apply this definition to all minor parties. Traditionally, third parties have served as venues of political dissent for voting individuals dissatisfied with the major candidates in an election. They often tackle specific issues otherwise ignored by major political parties, thus relinquishing popular support nationwide. As Supreme Court Justice Earl Warren wrote in 1957, "History has amply proved the virtue of political activity by minority, dissident groups, which innumerable times have been in the vanguard of democratic thought and whose programs were ultimately accepted" [37]. Hence, while third parties rarely capture the majority vote, their agendas, often incorporated into major party platforms, are significant nonetheless.

Given the potential relevance of third parties to national politics, we study, qualitatively and numerically, the dynamics of the emergence and spread of third parties on a local level where growth is measured in terms of the number of third party voters and members. We restrict our study to a local level because third parties usually originate in a small group and, via a "bottom-up" method of diffusion, spread within a population by acquiring local official positions and then expanding to higher levels of government [14]. Although individual personalities and circumstances dominate the initial formation of any group, the ability of even a small group of people to make itself heard within a larger community is great enough, especially in an age of information technology, that we consider the local level large enough to be described by a collective average. That is, the voting population under study is large enough that stochastic (random) effects are dominated by the deterministic average behavior of the group.

We use an epidemiological paradigm [2] to translate third party emergence from a political phenomenon to a mathematical one where we assume that third parties grow in a similar manner as epidemics in a population. We take this approach following in the steps of previous theoretical studies that model social issues via such methods [1, 6, 7, 11, 33]. The epidemiological metaphor is suggested by the assumption that individuals' decisions are influenced by the collective peer pressure generated by others' behavior; the "contacts" between these two groups' ideas are analogous to the contact processes that drive the spread of infectious diseases. Here we assume that a certain subpopulation of the voting individuals, defined according to certain demographic factors, is more receptive (in epidemiological terms, susceptible) to third party ideology than the rest of the voting population, and that their political behavior is therefore driven by such collective peer-pressure contacts [34, 36, 38].

There are many components that may affect the decision of a person when it comes to voting. During a political campaign all candidates spend a great amount of effort to assure that as many people as possible get exposed to their ideas, and most importantly, their name. People are often so overwhelmed by exposure to

different candidates that they end up depending more on informal ways of obtaining information such as talking to other people. This makes networking among people very influential during election time; in fact some studies show that voting is "contagious." [23] It has been shown that people often rely on friends, relatives, coworkers, etc. to obtain information about the candidates [31]. Also, there is evidence that people who support a particular candidate tend to encourage others to vote for that candidate [13]. This makes the interaction between people a very important factor in the voting process because both the person wanting information seeks it from other people and the ones with particular preferences want to share them.

Collective behaviors such as voting have been studied for decades, notably in the seminal work of Granovetter [12], who considered individuals in a population to have a distribution of thresholds with regard to their willingness to participate in a particular collective behavior (he used rioting as a primary example). Other subsequent studies considered the role played by a core group of especially influential individuals [15, 24], such as party activists recruiting voters. These studies used probabilistic and stochastic models which yielded primarily numerical results; in applying the epidemiological framework described above, we focus on the collective (rather than individual) thresholds for persistence of a behavior (here voting and party membership) which our deterministic models allow us to calculate.

While our model is designed to pertain to all third parties, we consider the Green Party as a case study. Although formally united under the Association of State Green Parties in 1996 (and later the nationalized Green Party of the United States in 2001), state-based green parties have thrived in the U.S. at the local level since 1984, when the Green Committees of Correspondence (CoC) were formed with the purpose of organizing local Green groups and working toward the founding of a national Green political party [18]. Our particular study focuses on the growth (and in some cases decline) of the Green Party in six states and the District of Columbia in the past decade, using voting and registration data. In comparing the predictions of our model to particular data, we consider a short time frame so that we can assume that social structure within the state in question does not change drastically, a necessary condition for assuming voting population heterogeneity.

We organize our paper as follows: Section 2 describes our theoretical model, as well as a simplification in which the entire population under study is equally receptive to third party ideas. Section 3 presents the mathematical analysis of the simplified model (which employs qualitative analysis techniques from the field of nonlinear dynamical systems, as well as sensitivity analysis) and interprets the results, and Section 4 applies them to a case study, using data to estimate model parameters. Section 5 performs a similar but more limited analysis of the more complex model, and Section 6 draws conclusions about the implications of our models for the growth of third parties from grassroots movements.

## 2. **A Population Model for the Spread of a Third Party.**

2.1. **Underlying Assumptions.** In developing our model, we apply epidemiological terminology to describe the growth of a third party. The assumptions we make about how individuals behave, and change their behavior, define the classes in our compartmental model and the rates at which individuals move between classes.

(1) We assume that our population is a heterogeneous mix of individuals who belong to different backgrounds according to certain demographic factors.

Our model considers a population of all voters, $N$, divided into two classes or sub-populations whose susceptibility to third party ideology is based on demographic factors such as education, socioeconomic status, race, gender, age, political orientation and professional occupation. Inherently, certain demographic characteristics, labelled as high affinity, make an individual more likely to subscribe to a third party's ideology, which targets a more specific audience than alternative majority agendas. That is, upon entering the voting system, certain individuals are more statistically inclined to vote a certain way. For example, a progressive environmental activist is statistically more likely to agree and vote for the Green Party agenda, which stresses communal-based economics, local government, and gender and racial equity, than a conservative corporate executive whose economic philosophy directly conflicts with that of the Green Party. For this reason we consider population heterogeneity vital to this study.

We apply the following method of dividing the entering voting population into two susceptible classes: if an individual has more high affinity factors than low affinity factors then that person directly enters the high affinity class and similarly for low affinity susceptibles. We define high affinity factors as features of the individual based on his/her demographic profile that make him/her more inclined to vote for the third party; conversely, low affinity factors make the individual less statistically likely to subscribe to the party's platform. We assume that a constant proportion $p$ $(0 < p < 1)$ of new voters enters the high-affinity class $H$ (the remaining $1 - p$ proportion enter $L$).

(2) We assume that individuals' affinity factors (demographic characteristics) remain fixed for the period of the study.
(3) We assume that a third party's agenda remains consistent over time.

We assume that individuals do not move from one susceptibility class to the other. One reason for this is the relative permanence of individuals' demographic characteristics. We limit our model to tracing the *expansion* of the third party; hence, we refer to a shorter time period over which we assume social structure remains constant. In other words, individuals with high affinity to the third party do not become individuals with low affinity, and vice versa.

The other reason for this consistency has to do with the parties themselves. In addition to being more specific than major party agendas (i.e., more specific in their goals and less geared to moderacy), third party platforms tend to be more consistent over time. Third parties are not pressured to constantly adjust to the shifting demands of the populace since they do not seek the majority vote. Consequently, they do not target the majority voting population. Each party has its own agenda, which appeals to certain sectors of the voting population. Hence different parties target voting populations that are more inclined to subscribe to their ideology. While one party, for example, may target individuals from a certain educational background that we, in our model, label as high affinity and that other parties may overlook, all parties nonetheless recognize that education factors into an individual's likelihood to support or refute that party's platform. It is true that individuals from varied backgrounds comprise the main parties, yet, when dealing with the specific agendas of third parties that do not strive to sway the majority vote, we assume that third parties appeal to individuals of certain demographic backgrounds more than others. Therefore, we account for the aforementioned standard set of demographic factors that parties look at when spreading their ideologies. In our paper we apply our model to an individual case study of the Green Party of Pennsylvania; however,

the same methodology of distinguishing susceptibles can be applied to all third parties.

(4) We define two levels of participation in third-party politics: voting for third-party candidates, and membership. A party exists only if it has members; we define members as those who pay dues, volunteer, and preside over party affairs.

As described above, all voters enter the voting system either to the low affinity, $L$, or high affinity, $H$, susceptible class. According to our epidemiological metaphor, in addition to these two susceptible classes, our model includes three infected classes: $V_H$, $V_L$, and $M$, third party voters from the high affinity class, third party voters from the low affinity class, and party members respectively. We define party members as voters of the third party who pay dues to the party; often such members officiate, volunteer and actively campaign for voter recruitment. In epidemiological terms, $V_H$ and $V_L$ correspond to voters of a lower degree of infection and individuals of the $M$ class are voters infected to a higher degree. We distinguish between $V_H$ and $V_L$ because of their interactions with their respective "neighbors" in $H$ and $L$.

(5) We assume that third parties, emerging through resource-limited grassroots efforts, spread primarily via primary (direct) contacts between third-party supporters and susceptibles.
(6) We assume that third party members have a greater effect upon voter recruitment than do third party voters, due to members' activism.

In our model, a system of nonlinear differential equations, we consider susceptible movement into voting and member compartments as well as possible regressions back from the third party voting phase into the susceptible class. Once an individual is susceptible he/she can become 'infected' (either $V_H$ or $V_L$) through direct contact with the $V_H$, $V_L$, and $M$ classes. Due to a lack of funding and resulting lack of mass-media exposure to the general population, these primary contacts with susceptibles involve such direct interaction as personal meetings, phone conversations, and electronic communications like personally addressed e-mails and weblog (blog) comments. We assume that the rate at which these contacts occur is proportional both to the size of the susceptible group and to the proportion of third-party supporters in the population.

We do not consider a linear term weighing the influence of media coverage from the third party (i.e., secondary contact factors) in the forward transition from both susceptible classes to third party voting classes. Instead, we focus on the nonlinear terms considering the effects of voters from the $V_H$, $V_L$, and $M$ classes, where voters from $V_H$ and $V_L$ bear an affinity-specific influence $\beta_H$ (from $H$ to $V_H$) and $\beta_L$ (from $L$ to $V_L$) in third party voter recruitment. Through activism, members from $M$ influence susceptibles of each type at higher rates ($\alpha\beta_H$ and $\alpha\beta_L$, respectively) than voters, with their increased influence measured by the parameter $\alpha$ ($\alpha > 1$).

(7) We consider both primary (direct) contacts and secondary (indirect) contacts in the regression of third party voters to the susceptible class.
(8) We assume that all other parties exert equal influence in discouraging third party voting.
(9) We assume that individuals have more influence upon others of their same affinity class (high or low) than upon members of the other affinity class, in encouraging and discouraging third party voting.

We consider the transitions back from the third party voting to the suscepti-
ble classes to involve both linear terms, $\epsilon_H V_H$ and $\epsilon_L V_L$, and nonlinear terms,
$\phi_H(H + \sigma L)\frac{V_H}{N}$ and $\phi_L(L + \sigma H)\frac{V_L}{N}$, contributions by secondary contacts with the
opposition (i.e., media from well-funded majority voters) and direct contact with
the susceptible classes respectively. Compared to primary contacts, described in a
previous paragraph, secondary contacts include mass e-mails, media, and circulating
literature.

Voters from a certain susceptibility class (with its own set of demographic fac-
tors) address issues that usually appeal more to voters deriving from the same
class. Therefore, susceptible individuals with high affinity bear a greater influence
in recruiting voters who came from the high affinity susceptible class back into the
susceptible class than susceptible individuals with low affinity. The reduction in
influence by individuals from a different affinity class is denoted by $\sigma$, so that in
regressing back from $V_H$ to $H$, $\sigma L$ represents the lesser influence that $L$ individu-
als exert on voters from $V_H$ than do susceptibles from $H$, the higher affinity class
(similar reasoning applies to the $V_L$-to-$L$ transition). Likewise, the cross-affinity
influence in recruiting third party voters is reduced by a factor of $\sigma$.

(10) We assume that third party voters become active party members through
the ongoing efforts (primary contacts) of the members, who have made a
permanent commitment to the party.

Once voting for the third party, individuals can become party members. They
enter this higher state of infection via the nonlinear terms $\gamma V_H \frac{M}{N}$ and $\gamma V_L \frac{M}{N}$,
where we only consider the influence of primary contacts with party members in
bringing about this transition, measured by the rate parameter $\gamma$. Given that we
are studying the spread of the party, we assume that party members do not resign
their memberships. We reason that once an individual feels strongly enough to join
a party, he/she retains his/her loyalty to the party; the only way a person stops
being a member (during the growth period under study) is by leaving the voting
system.

Finally, we consider natural exits from all classes (at a rate $\mu$) as a result of death
or moving. The sum of the equations of the model, for both versions developed
below, gives $\frac{dN}{dt} = 0$, reflecting an assumption that the total voting population size
remains constant, i.e., the number of people entering the voting system (coming of
age, or moving in) counterbalances the number of people leaving the system (dying,
or moving out), which is a reasonable approximation for periods of a few years.

2.2. **The General (Two-Track) Model.** We first introduce a two-track model,
as described immediately above, to study the dynamics between a heterogeneously
mixed population of susceptible voters, third party voters, and party members. We
apply the following set of ordinary differential equations to model voting dynamics,
as illustrated in Figure 1.

$$\frac{dH}{dt} = p\mu N + \epsilon_H V_H + \phi_H(H + \sigma L)\frac{V_H}{N} - \beta_H(V_H + \sigma V_L + \alpha M)\frac{H}{N} - \mu H, \qquad (1)$$

$$\frac{dL}{dt} = (1 - p)\mu N + \epsilon_L V_L + \phi_L(L + \sigma H)\frac{V_L}{N} - \beta_L(\sigma V_H + V_L + \alpha M)\frac{L}{N} - \mu L, \quad (2)$$
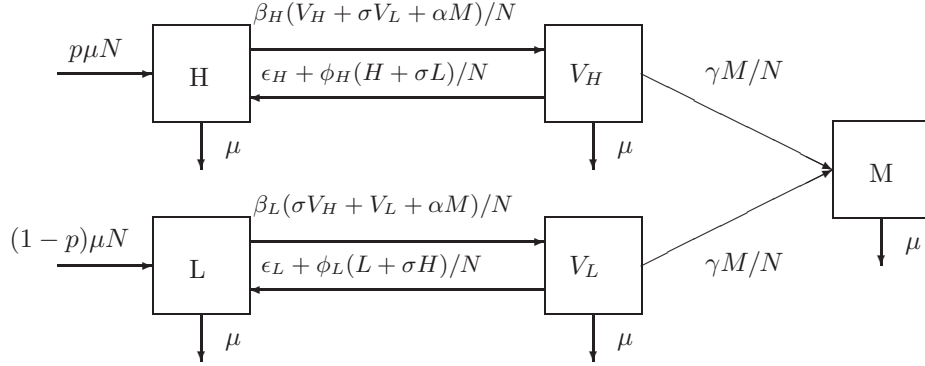
FIGURE 1. The two-track model, with per capita flow rates

Table of Compartments and Parameters

| | |
|---|---|
| $H$ | high affinity susceptibles (i.e., voters highly susc. to third party ideology) |
| $L$ | low affinity susceptibles (i.e., voters barely susc. to third party ideology) |
| $V_H$ | third party voting individuals deriving from $H$ |
| $V_L$ | third party voter individuals deriving from $L$ |
| $M$ | third party members (i.e., party officials, donors, volunteers) |
| $p$ | proportion of the voting population $N$ entering $H$ |
| $\beta_H$ | peer-driven recruitment rate of $H$ into $V_H$ by individuals in $V_H$, $V_L$ and $M$ |
| $\epsilon_H$ | linear recruitment rate of $V_H$ back into $H$ via secondary contacts (i.e., media and campaigning from opposing parties) |
| $\phi_H$ | recruitment rate of $V_H$ into $H$ by direct contact with individuals in the opposition classes (i.e., individuals in H and L) |
| $\beta_L$ | peer driven recruitment rate of $L$ into $V_L$ by individuals in $V_L$, $V_H$ and $M$ |
| $\epsilon_L$ | linear recruitment rate of $V_L$ back into $L$ via secondary contacts (i.e., media and campaigning from opposing parties) |
| $\phi_L$ | recruitment rate of $V_L$ into $L$ by direct contact with individuals in the opposition classes (i.e., individuals in $H$ and $L$) |
| $\alpha$ | factor by which the influence of party members $M$ in recruiting voters in $H$ and $L$ into $V_H$ and $V_L$ exceeds that of voters in $V_H$ and $V_L$ |
| $\sigma$ | factor by which the influence of individuals upon members of a different affinity class is reduced in encouraging or discouraging third party voting |
| $\gamma$ | recruitment rate of $V_H$ and $V_L$ into $M$ by individuals in $M$ |
| $\mu$ | rate at which individuals enter or leave the voting system |

TABLE 1. Compartments and parameters of the two-track model

$$\frac{dV_H}{dt} = \beta_H(V_H + \sigma V_L + \alpha M)\frac{H}{N} - \epsilon_H V_H - \phi_H(H + \sigma L)\frac{V_H}{N} - \frac{\gamma M V_H}{N} - \mu V_H, \quad (3)$$

$$\frac{dV_L}{dt} = \beta_L(\sigma V_H + V_L + \alpha M)\frac{L}{N} - \epsilon_L V_L - \phi_L(L + \sigma H)\frac{V_L}{N} - \frac{\gamma M V_L}{N} - \mu V_L, \quad (4)$$

$$\frac{dM}{dt} = \frac{\gamma M V_H}{N} + \frac{\gamma M V_L}{N} - \mu M, \quad (5)$$

$$N = H + L + V_H + V_L + M. \quad (6)$$

Adding equations (1), (2), (3), (4) and (5) yields $\frac{dN}{dt} = 0$, showing that the total population $N$ is constant over time. Model parameters are summarized in Table 1.

2.3. **The Simplified (One-Track) Model.** In order to facilitate analysis of the two-track model, we initially consider a simplified version that does away with voting population heterogeneity (assumption (1), and consequently (9)) before exploring analysis for the more complex system. This simplified model assumes a homogeneous susceptible population ($p = 0$ or $p = 1$), reducing the two-track model to one susceptible class, $S$, and two infected classes: third party voters, $V$, and party members, $M$, respectively. The $S$ class comprises those individuals who vote, but do not vote for the third party. The $V$ class comprises the third party voters, and the $M$ class again has third party members (i.e., party officials, donors, volunteers).

In the one-track model we omit unnecessary parameters from the heterogeneous version. Figure 2 illustrates the one-track model, and Table 2 summarizes the parameters.
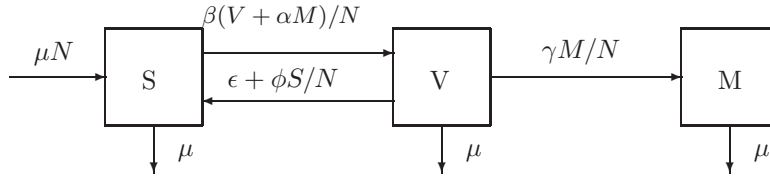


FIGURE 2. The one-track model, with per capita flow rates

| | |
|---|---|
| $\beta$ | peer driven recruitment rate of $S$ into $V$ by third party voters and members |
| $\epsilon$ | recruitment rate of $V$ back into $S$ via secondary contacts (i.e., media and campaigning from opposing parties) |
| $\phi$ | recruitment rate of $V$ into $S$ by direct contact with susceptibles |
| $\alpha$ | factor by which the recruitment rate of $S$ into $V$ by third party members exceeds the recruitment rate by individuals in $V$ |
| $\gamma$ | recruitment rate of $V$ into $M$ by third party members |
| $\mu$ | rate at which individuals enter or leave the voting system |

TABLE 2. Parameters of the one-track model

In this case the model reduces to the following system, which is effectively two-dimensional since $N$ can again be seen to be constant:

$$\frac{dS}{dt} = \mu N + \epsilon V + \phi S \frac{V}{N} - \beta(V + \alpha M)\frac{S}{N} - \mu S, \tag{7}$$

$$\frac{dV}{dt} = \beta(V + \alpha M)\frac{S}{N} - \epsilon V - \phi S \frac{V}{N} - \frac{\gamma MV}{N} - \mu V, \tag{8}$$

$$\frac{dM}{dt} = \frac{\gamma MV}{N} - \mu M, \tag{9}$$

$$N = S + V + M. \tag{10}$$

3. **Analysis of the One-Track Model.** We begin our analysis by calculating equilibria for our model and determining conditions for their existence and stability. We first simplify the system in two ways. Since the total voting population $N$ is assumed constant, we can reduce our system to two dimensions by rewriting $S = N - V - M$ in equation (8), so that

$$\frac{dV}{dt} = \beta(V + \alpha M)\frac{N - V - M}{N} - \epsilon V - \phi(N - V - M)\frac{V}{N} - \frac{\gamma MV}{N} - \mu V. \quad (11)$$

We can now analyze the system defined by (11) and (9), as $S$ can always be found once $V$ and $M$ are known.

The constancy of $N$ also allows us to proportionalize the system by defining new variables $v = \frac{V}{N}$, $m = \frac{M}{N}$, and $s = \frac{S}{N} = 1 - v - m$, which give the proportion of the population in each class. Dividing equations (11) and (9) by $N$ and substituting the new variables gives, finally, the system

$$\frac{dv}{dt} = \beta(v + \alpha m)(1 - v - m) - \epsilon v - \phi(1 - v - m)v - \gamma mv - \mu v, \quad (12)$$

$$\frac{dm}{dt} = (\gamma v - \mu)m. \quad (13)$$

In order to analyze stability we linearize the system and compute partial first derivatives with respect to each of the variables, $v$ and $m$, obtaining the Jacobian matrix $J_1$ for system (12)–(13):

$$J_1 = \begin{pmatrix} (\beta - \phi)(1 - 2v - m) & \alpha\beta(1 - v - 2m) - (\beta - \phi + \gamma)v \\ -(\alpha\beta + \gamma)m - (\mu + \epsilon) & \\ \gamma m & \gamma v - \mu \end{pmatrix}.$$

3.1. $E_1$**: Party-Free Equilibrium (PFE).** The party-free equilibrium (PFE) for the reduced system occurs at (0,0), the steady state achieved when the entire population resides in the $S$ class (i.e., the third party has neither voters nor members and, by definition of party existence, does not exist). The PFE is essentially analogous to the disease-free equilibrium in epidemiology and always exists as a possible outcome for the voting population.

Applying the above reduced Jacobian matrix to our PFE, (0,0), where we only consider the $v$ and $m$ terms, we determine PFE stability:

$$J_1(0,0) = \begin{pmatrix} (\beta - \phi) - (\mu + \epsilon) & \alpha\beta \\ 0 & -\mu \end{pmatrix}.$$

The equilibrium point (0,0) will be locally asymptotically stable (LAS) if all the eigenvalues of the matrix are negative. Assuming $\mu > 0$, the eigenvalue $-\mu$ of the Jacobian is always negative, whereas the second eigenvalue $(\beta - \phi) - (\mu + \epsilon) < 0$ if and only if $\frac{(\beta - \phi)}{\mu + \epsilon} < 1$. For ease of notation and interpretation we define the threshold quantity $R_1 = \frac{(\beta - \phi)}{\mu + \epsilon}$, so that the PFE is LAS if and only if $R_1 < 1$. We discuss the relevance of this threshold value in the last part of this section.

3.2. $E_2$**: Member-Free Equilibrium (MFE).** The member-free equilibrium (MFE) occurs when $M = 0$ but $V, S \neq 0$, i.e., the voting population subdivides between susceptibles, $S$, and third party voters, $V$. While mathematically possible, this outcome is politically unrealistic given that voters cannot vote for a party that does not exist (recall our assumption (4) that party existence depends on the

presence of an $M$ class). For mathematical consistency, however, we consider the equilibrium point $(v_2^*, m_2^*)$, where $m_2^* = 0$. (Note that the asterisk superscript denotes equilibrium values, while the numerical subscript distinguishes the equilibrium point in question.) This arises from the equilibrium condition obtained by setting $dm/dt = 0$ in (13):

$$(\gamma v^* - \mu)m^* = 0, \tag{14}$$

which implies that either $(\gamma v^* - \mu) = 0$ (which we will consider later) or $m^* = 0$.

To find $v_2^*$ we set $dv/dt = 0$ and $m^* = 0$ in (12) and rearrange terms to get

$$(\beta - \phi)v^{*2} + (\mu + \epsilon + \phi - \beta)v^* = 0.$$

This implies that either $v^* = 0$ (the party-free equilibrium) or $(\beta - \phi)v^* = \beta - (\mu + \epsilon + \phi)$. We consider the situation where $v^* \neq 0$, solve for $v^*$ and simplify the results as follows:

$$v_2^* = 1 - \frac{\mu + \epsilon}{\beta - \phi} = 1 - \frac{1}{R_1}$$

where $v_2^*$ retains political value only if $R_1 > 1$—otherwise $v_2^* < 0$ which is meaningless.

Finally, we can also write $s_2^* = 1 - v_2^* - m_2^* = \frac{\mu + \epsilon}{\beta - \phi} = \frac{1}{R_1}$, which makes sense politically only if $R_1 > 1$ (i.e, $s_2^* < 1$). We can therefore express our member-free equilibrium as $E_2 = (1 - \frac{1}{R_1}, 0)$. The MFE exists if and only if $R_1 > 1$, since ignoring this condition leads to a negative third party voting population.

The above situation makes mathematical sense but not political sense since parties, by our original assumption, do not exist without members, and in this member-free case we deal with voters who vote for a non-existent party. We might interpret this situation as having voters still willing to vote for this party, but no party candidates for whom to vote.

Regardless of the political likelihood of MFE existence, we consider its stability. Again, we apply the method of using the reduced system's Jacobian matrix in determining the stability of the member-free equilibrium:

$$J_1(1 - \tfrac{1}{R_1}, 0) = \begin{pmatrix} (\mu + \epsilon)(1 - R_1) & \alpha\beta(\frac{1}{R_1}) - (\beta - \phi + \gamma)(1 - \frac{1}{R_1}) \\ 0 & \gamma(1 - \frac{1}{R_1}) - \mu \end{pmatrix}.$$

The reduced system equilibrium point $(1 - \frac{1}{R_1}, 0)$ is locally asymptotically stable if all the eigenvalues of the above matrix are negative. We know that, since $R_1 > 1$ (in order for the MFE to exist), one of the eigenvalues, $(\mu + \epsilon)(1 - R_1)$, is negative. The second eigenvalue of the Jacobian is $\gamma(1 - \frac{1}{R_1}) - \mu$. This eigenvalue is negative if and only if $\frac{\gamma}{\mu}(1 - \frac{1}{R_1}) < 1$.

We define the left hand side of the inequality as $R_2 = \frac{\gamma}{\mu}(1 - \frac{\mu + \epsilon}{\beta - \phi}) = \frac{\gamma}{\mu}(1 - \frac{1}{R_1})$. Hence we have derived two threshold parameters $R_1$ and $R_2$ that determine equilibria stability depending on relative parameter values. (Note that they are related: $R_1 = 1 \Leftrightarrow R_2 = 0$.)

### 3.3. $E_3$ and $E_4$: Survival Equilibria.

In the event of survival equilibria, the voting population subdivides between susceptibles, $S$, third party voters, $V$, and members, $M$. We regard this as a successful state of coexistence and, given certain conditions, the point at which the party thrives. We determine the equilibrium proportions by returning to the condition (14) that $dm/dt = 0$. Since $\frac{dm}{dt} = (\gamma v^* -$

$\mu)m^*$, $v^* = \frac{\mu}{\gamma}$ when $m^* \neq 0$. Here we impose the condition $\mu < \gamma$ so that $v^* < 1$, since $s^* + v^* + m^* = 1$.

Next we set $dv/dt = 0$ and $v^* = \frac{\mu}{\gamma}$ in (12):

$$\frac{dv}{dt} = (\beta - \phi)\frac{\mu}{\gamma}\left(1 - \frac{\mu}{\gamma} - m^*\right) + \alpha\beta m^*\left(1 - \frac{\mu}{\gamma} - m^*\right) - (\mu + \epsilon + \gamma m^*)\frac{\mu}{\gamma} = 0.$$

This equation, which is quadratic in $m^*$, can be rewritten (after dividing through by $-\alpha\beta$) in the form $f(m^*) = m^{*2} + BM^* + C = 0$, where

$$B = \left(\frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - \left(1 - \frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta}, \quad C = -\frac{\mu}{\gamma}\left[\frac{\beta - \phi}{\alpha\beta}\left(1 - \frac{\mu}{\gamma}\right) - \frac{\mu + \epsilon}{\alpha\beta}\right]. \quad (15)$$

Solutions to $f(m^*) = 0$ are given by the quadratic formula

$$m^*_\pm = \frac{1}{2}\left[-B \pm \sqrt{B^2 - 4C}\right];$$

however, in general, these solutions may be real, or may fall outside the meaningful interval $(0, 1 - v^*]$. Depending on the values of model parameters, there may be 0, 1, or 2 solutions within this interval, each corresponding to a meaningful equilibrium with positive party membership.

Analysis of the conditions involved in determining the number of survival equilibria is considerably more involved than that for $E_1$ and $E_2$; details are given in the Appendix. The results, which introduce an additional threshold quantity, can be summarized as follows.

**Proposition 1.** *(i) If $R_2 > 1$, the system (12)–(13) has precisely one survival equilibrium $E_3 = (\frac{\gamma}{\mu}, m^*_+)$.*

*(ii) If $R_2 < 1$, then the system has two survival equilibria, $E_3$ and $E_4 = (\frac{\gamma}{\mu}, m^*_-)$, if and only if $R_3 > 1$, where*

$$R_3 = \min\left(R_{3a}, R_{3b}\right), \quad R_{3a} = r_3\left(1 - \frac{1+q}{r_2}\right), \quad R_{3b} = \sqrt{r_3}\left(1 - \sqrt{\frac{1-q}{r_2} + h}\right), \quad (16)$$

*and*

$$q = \frac{\beta - \phi}{\alpha\beta}, \quad r_2 = \gamma/\mu, \quad r_3 = \alpha\beta/\mu, \quad h = \frac{2}{\sqrt{r_3}}\left(\sqrt{1 + \frac{\epsilon}{\gamma}} - 1\right).$$

*Otherwise there are none.*

To interpret the conditions $R_{3a} > 1$ and $R_{3b} > 1$, we can rewrite them as follows:

$$B < 0 \Leftrightarrow R_{3a} > 1 \Leftrightarrow \frac{1}{\alpha\beta} + \frac{1+q}{\gamma} < \frac{1}{\mu}, \quad (17)$$

$$B^2 - 4C \geq 0 \Leftrightarrow R_{3b} > 1 \Leftrightarrow \sqrt{\frac{1}{\alpha\beta}} + \sqrt{\frac{1-q}{\gamma} + \hat{h}(\epsilon/\gamma)} \leq \sqrt{\frac{1}{\mu}}, \quad (18)$$

where

$$\hat{h}(\epsilon/\gamma) = 2\sqrt{\frac{1}{\alpha\beta}}\sqrt{\frac{1}{\mu}}\left(\sqrt{1 + \frac{\epsilon}{\gamma}} - 1\right) > 0.$$

Both inequalities (17) and (18) relate the minimum amounts of time taken for an individual to be influenced by a party member ($M$) to move from $S$ to $V$, $1/\alpha\beta$, and from $V$ to $M$, $1/\gamma$, to the average lifetime of an individual in the voting system, $1/\mu$. In order for the party to survive, a weighted sum of the first two times must be less than the average lifetime in the system, or else individuals in $S$ will, on average,

leave the system before they can "replace" the members in $M$ who recruited them. The weights in the sum involve $q = \frac{\beta - \phi}{\alpha\beta}$, the relative effectiveness of voters $V$ to members $M$ in recruiting new voters from $S$, because the influence of party voters reduces the recruiting threshold burden on party members $M$ to some extent.[1] (Note that $0 < q < 1$.)

Observe that (18) implies that

$$\sqrt{\frac{1}{\alpha\beta}} + \sqrt{\frac{1-q}{\gamma}} \leq \sqrt{\frac{1}{\mu}}. \tag{19}$$

If $\epsilon << \gamma$, we can generate a Maclaurin (Taylor) expansion in $\epsilon/\gamma$ for (18) which yields

$$\sqrt{\frac{1}{\alpha\beta}} + \sqrt{\frac{1-q}{\gamma}} + \frac{1}{2} \frac{\sqrt{\frac{1}{\alpha\beta}}\sqrt{\frac{1}{\mu}}}{\sqrt{\frac{1-q}{\gamma}}} \left(\frac{\epsilon}{\gamma}\right) + \mathcal{O}\left[\left(\frac{\epsilon}{\gamma}\right)^2\right] \leq \sqrt{\frac{1}{\mu}}.$$

This expansion suggests more quantitatively how the influence $\epsilon$ of voters for other parties $S$ in getting third-party voters $V$ to "defect" back to the major parties complicates the third party's survival (which would otherwise only require (19)).

Having established conditions for the existence of the two survival equilibria, $E_3$ and $E_4$, we examine their stability. The reduced Jacobian matrix for the survival equilibria follows:

$$J_1(\tfrac{\mu}{\gamma}, m_\pm^*) = \begin{pmatrix} \begin{bmatrix} (\beta - \phi)\left(1 - 2\frac{\mu}{\gamma} - m_\pm^*\right) \\ -(\alpha\beta + \gamma)m_\pm^* - (\mu + \epsilon) \end{bmatrix} & \begin{bmatrix} \alpha\beta\left(1 - \frac{\mu}{\gamma} - 2m_\pm^*\right) \\ -(\beta - \phi)\frac{\mu}{\gamma} - \mu \end{bmatrix} \\ \\ \gamma m_\pm^* & 0 \end{pmatrix}.$$

The stability (LAS) criterion that the eigenvalues of this matrix have negative real part is equivalent to the conditions that the trace be negative and the determinant positive. We calculate

$$\det J_1(\tfrac{\mu}{\gamma}, m_\pm^*) = -\gamma m_\pm^* \left[\alpha\beta\left(1 - \frac{\mu}{\gamma} - 2m_\pm^*\right) - (\beta - \phi)\frac{\mu}{\gamma} - \mu\right],$$

so that

$$\det J_1(\tfrac{\mu}{\gamma}, m_\pm^*) > 0 \Leftrightarrow \left[\alpha\beta\left(1 - \frac{\mu}{\gamma} - 2m_\pm^*\right) - (\beta - \phi)\frac{\mu}{\gamma} - \mu\right] < 0$$

$$\Leftrightarrow m_\pm^* > \frac{1}{2}\left[\left(1 - \frac{\mu}{\gamma}\right) - \left(\frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - \frac{\mu}{\alpha\beta}\right] = -\frac{B}{2}.$$

Since $m_-^* < -\frac{B}{2} < m_+^*$, we see that $E_4$ is never stable, while the stability condition for $E_3$ reduces to $\operatorname{tr} J_1(\tfrac{\mu}{\gamma}, m_+^*) < 0$. Thus we calculate

$$\operatorname{tr} J_1(\tfrac{\mu}{\gamma}, m_+^*) = (\beta - \phi)\left(1 - 2\frac{\mu}{\gamma}\right) - (\mu + \epsilon) - (\beta - \phi + \alpha\beta + \gamma)m_+^*,$$

so that

$$\operatorname{tr} J_1(\tfrac{\mu}{\gamma}, m_+^*) < 0 \Leftrightarrow m_+^* > \frac{(\beta - \phi)\left(1 - 2\frac{\mu}{\gamma}\right) - (\mu + \epsilon)}{(\beta - \phi + \alpha\beta + \gamma)} = \frac{(\beta - \phi)\frac{\mu}{\gamma}(R_2 - 2)}{(\beta - \phi) + \alpha\beta + \gamma}. \tag{20}$$

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[1] If we define $T_1 = 1/\alpha\beta$, $T_2 = 1/\gamma$, and $T_3 = 1/\mu$, (17) and (18) can be written more simply as $T_1 + (1 + q)T_2 < T_3$, $\sqrt{T_1} + \sqrt{(1-q)T_2 + h} \leq \sqrt{T_3}$.

| Equilibrium | Existence Cond. | Stability Cond. |
|---|---|---|
| Party-Free $E_1 = (1,0,0)$ | always exists | $R_1 < 1$ |
| Member-Free $E_2 = (\frac{1}{R_1}, 1 - \frac{1}{R_1}, 0)$ | $R_1 > 1$ | $R_2 < 1$ |
| Survival $E_3 = (1 - \frac{\mu}{\gamma} - m_+^*, \frac{\mu}{\gamma}, m_+^*)$ | (i) $R_2 > 1$, or | always stable |
| | (ii) $R_2 < 1$, $R_3 > 1$ | when it exists |
| Survival $E_4 = (1 - \frac{\mu}{\gamma} - m_-^*, \frac{\mu}{\gamma}, m_-^*)$ | $R_2 < 1$, $R_3 > 1$ | always unstable |

TABLE 3. Equilibria of one-track model

This is true for all $m^*$ when $R_2 < 2$. The case when $R_2 \geq 2$ requires further algebra and is relegated to the Appendix. The result is that $E_3$ is always stable (LAS) when it exists, while $E_4$ is always unstable.

3.4. **Global Behavior.** Table 3 summarizes the conditions for existence and local stability of the four equilibria of the one-track model. Because system (7)–(9) can be reduced to a set of two differential equations (12)–(13), we can apply the Poincaré-Bendixson Theorem to establish global stability. Since the system is well-posed, with the invariant set $D = \{(v, m) : v, m > 0;\ v + m \leq 1\}$, there are no unbounded solutions beginning in the state space, and a straightforward application of Dulac's Criterion with the function $b = 1/vm$ confirms that there are no limit cycles, either:

$$\frac{\partial}{\partial v}\left(b\frac{dv}{dt}\right) + \frac{\partial}{\partial m}\left(b\frac{dm}{dt}\right) < 0 \ \text{ in } D.$$

Therefore, all solutions to the system which begin within the state space must approach an equilibrium. In particular, when only one locally stable (LAS) equilibrium exists, that equilibrium must in fact be globally stable (GAS).

The conditions in Table 3 can be graphed to show the different possible global behaviors of the one-track model. Each condition corresponds to a curve dividing parameter space into multiple regions, each of which represents a different global behavior (see Appendix A.1 for the derivations). The five resulting regions are illustrated in Figure 3 and summarized in Table 4.

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ |
|---|---|---|---|---|
| I | stable | unstable | does not exist | does not exist |
| II | unstable | stable | does not exist | does not exist |
| III | unstable | unstable | stable | does not exist |
| IV | unstable | stable | stable | unstable |
| V | stable | does not exist | stable | unstable |

TABLE 4. Regions of Equilibrium Stability

  I In region I, $E_1$ is the only stable equilibrium; the party will always go extinct.

  II In region II where $R_1 > 1$, $R_2 < 1$, and $R_3 < 1$, $E_2$, the member-free state, is the only stable equilibrium.

 III In region III where both $R_1 > 1$ and $R_2 > 1$, $E_3$ is the only stable equilibrium; the party will inevitably approach a survival state.
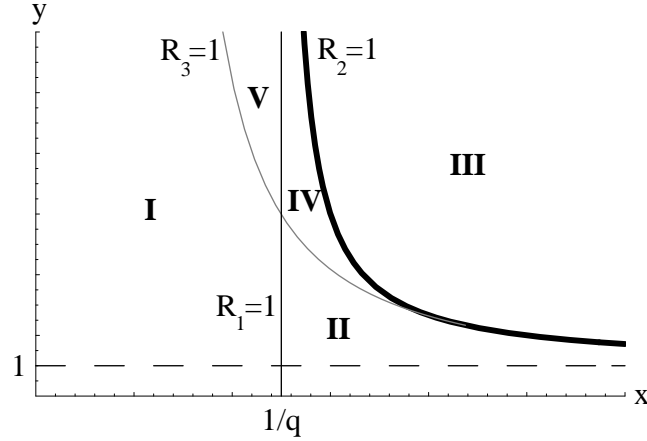
FIGURE 3. Regions of equilibrium stability, with $q$, $x$, $y$ as in Appendix A

IV In region IV where $R_1 > 1$, $R_2 < 1$ and $R_3 > 1$, $E_2$ and $E_3$ coexist as stable equilibria although, if placed in a political context, $E_2$ is not a realistic outcome for the party.

V In region V where $R_1 < 1$, $R_2 < 1$, and $R_3 > 1$, $E_1$ and $E_3$ are both stable; depending on the initial conditions the solution tends to one state or the other.

The coexistence of two locally stable equilibria (one representing party survival and the other party extinction) in regions IV and V involves a phenomenon known in epidemic modeling as a backward bifurcation, in which the exchange of stability at a threshold value (here, $R_2 = 1$) reverses direction, creating a situation in which the unstable equilibrium $E_4$ serves to split the state space into two basins of attraction for the two stable equilibria. In these regions, the equilibrium approached depends upon initial conditions: in particular, a large enough core of dedicated members $M$ can sustain the party (toward $E_3$). What is unusual here is that the "backward" part of the bifurcation curve can extend back beyond not only the bifurcation at $R_2 = 1$ but also the bifurcation at $R_1 = 1$ (where $R_2 = 0$), as illustrated in Figure 4. Whenever $R_1 < 1$, $R_2 < 0$, a condition that would normally lead to the death of the party given local asymptotic stability of the PFE, there are still conditions ($R_3 > 1$) under which two survival equilibria exist, one of which is stable ($E_3$). In other words, the party can thrive in conditions under which it would normally die out, given that we have the necessary parameters and sufficient initial number of $M$ individuals.

The ideal conditions for the party, of course, are when $R_2 > 1$ (region III), in which the party survives regardless of initial conditions. (Note $R_2 > 1$ implies $R_1 > 1$.) The last part of this section interprets all these mathematical thresholds in political terms.

3.5. **Threshold Parameters $R_1$, $R_2$, and $R_3$.** Our system contains three local thresholds or tipping points where population outcomes, measured as $S$, $V$, and $M$, depend on parameter values. By tipping point we refer to the sociological term that describes the point at which a stable phenomenon turns into a crisis, which, in a political context, corresponds to the extreme states of the party: death and growth [10]. In the context of our model, for example, the party cannot sustain
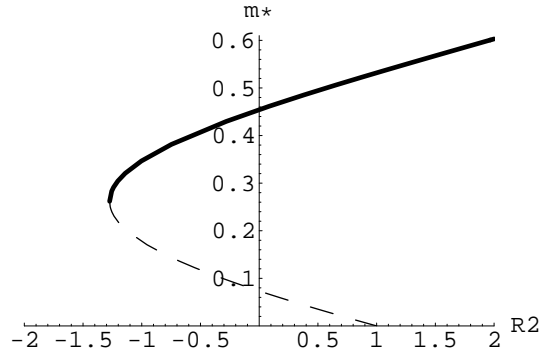
FIGURE 4. Diagram illustrating a backward bifurcation that extends below $R_2 = 0$ ($R_1 = 1$), using parameter values $\phi = 0.15$, $\mu = 0.05$, $\epsilon = 0.25$, $\gamma = 0.20$ (all rates in $\mathrm{yr}^{-1}$), $\alpha = 1.25$

grassroots growth until parameter conditions reach $R_1 = 1$, after which point the third party voting and member classes gain individuals. These threshold quantities correspond to similar terms in demographic and epidemic models called *reproductive numbers* which measure the average number of offspring or infections caused by a single group member during its lifetime. In our political context it is more appropriate to interpret these reproductive numbers in collective terms as regards the influence of party voters on the susceptible population. We distinguish between the aforementioned thresholds, $R_1$, $R_2$, and $R_3$, by analyzing them qualitatively in a political context.

$R_1$ denotes the average number of susceptibles influenced to vote for the party by a single party voter in $V$, if dropped into a homogeneous population of susceptibles. The expression $R_1 = \frac{\beta - \phi}{\mu + \epsilon}$ gives the net peer pressure, $\beta - \phi$, on susceptibles $S$ by voters $V$, multiplied by the average time, $\frac{1}{\mu + \epsilon}$, spent in the voting class $V$. As mentioned above, it is most appropriate to interpret $R_1$ as an average number per voter taking into account the collective influence of all the party voters. Note that we are assuming that the influence of party voters on susceptibles $\beta$ is stronger than the reverse influence $\phi$ of susceptibles to discourage party voters, a necessary condition for party emergence, which guarantees that $R_1 > 0$.

$R_2$ measures instead the second stage of recruitment: how effective party members $M$ are in recruiting third party voters to become members once there are enough individuals in $V$. The expression $R_2 = \frac{\gamma}{\mu}(1 - \frac{1}{R_1})$ gives the product of the rate at which party activists $M$ recruit voters from $V$ into $M$, $\gamma$, the average political lifetime of a party member, $\frac{1}{\mu}$, and the proportion of the population $N$ in $V$ at the MFE, $(1 - \frac{1}{R_1})$ (since only voters $V$ can be recruited directly into $M$). Similar to $R_1$, $R_2$ measures the average number of $V$ to $M$ conversions per individual in the $M$ class. If party voters are ineffective at influencing susceptibles to vote for the party and $R_1 < 1$, then the value of $R_2$ will be negative; rather than interpreting this as a negative ability of party members to recruit voters into activism, however, it should be seen as an indication that the pool of party voters available for recruitment into party membership is not there, preventing normal party growth.

$R_3$ measures the extent to which party members $M$ actively recruit susceptible individuals $S$ into the voting class $V$. This activism, which sidesteps the traditional

hierarchical structure of a party, is a key characteristic of growing grassroots movements, which often lack a political environment favorable to their growth in the traditional way outlined above ($R_2 < 1$). Because of the two conditions required mathematically for party survival when $R_2 < 1$, $R_3$ is defined as the greater of two quantities, both of which involve party members' abilities to recruit voters, $r_2$, and members, $r_3$, over their political lifetimes, as well as the relative efficacy of voter peer-pressure influence to members' influence through activism on the susceptible, $q$. The general form $r_3(1 - 1/r_2)$ present in both components of $R_3$ parallels the form of $R_2 = r_2(1 - 1/R_1)$, with members' potential $r_2$ for converting third-party voters into members reducing the constraint on their ability $r_3$ to produce those voters in the first place. Note that $r_2$ must be great enough in order for $R_3$ even to be positive: that is, only if the members can recruit other members well can their recruitment of voters sustain the party.

We now list possible outcomes involving these thresholds and their implications:

1. When $R_1 < 1$ (regions I and V in Figure 3 and Table 4) the net influence of party voters $V$ upon susceptibles is weak enough that grassroots emergence of the party is not possible. The hierarchical structure of party involvement precludes normal growth of party membership $M$ when party voters are unable to replenish their own ranks ($R_1 < 1$ implies $R_2 < 0$). However, exceptionally, if the recruiting ability of party members $M$ is great enough ($R_3 > 1$, region V), a sufficiently large dedicated core can sustain the party through its own efforts, despite the relative inefficiency or lack of influence of those merely voting for the party. This outcome illustrates the key role activists play in party survival during periods of adversity.

2. When $R_1 > 1$ each individual in $V$ is converting, on average, more than one person in $S$ into $V$, thus allowing the voting class $V$ to thrive. If, in addition, $R_2 < 1$ (regions II and IV), we have a situation in which party activists are then unable to recruit from the voting class effectively enough to maintain the party core $M$ (perhaps because the voting class is too small). Normally this would lead to an outcome in which a group disposed to vote for the party remains (in $V$) but the party core itself dwindles away, leading effectively to party extinction (with the ideologically closest main political party perhaps adjusting a platform to capture these votes). However, as with the case when $R_1 < 1$, a sufficiently large initial core can ensure the party's survival when the core is effective at influencing individuals to begin voting for the party ($R_3 > 1$, region IV).

3. The condition $R_2 > 1$ explains the case where the party ($V$ and $M$ classes) grows normally, by recruiting members from the $S$ and $V$ populations, respectively. Here the party voters are influential enough in garnering new voters from $S$ ($R_1 > 1$) that the party's survival does not depend on party activists' ability to recruit new voters directly (i.e., $R_3$ does not come into play). In epidemiological terms, this corresponds to a successful invasion: conditions are so favorable for the development of the party that it will become established even with a small initial group of members.

In general, the survival of the party is determined by the interplay among the three recruitment processes involved in the model: from $S$ to $V$ by members of $V$, as measured by $R_1$; from $V$ to $M$ by members of $M$, as measured by $r_2$ ($R_2$ incorporates both of these first two processes); and from $S$ to $V$ by members of $M$, as measured by $r_3$ ($R_3$ incorporates all three processes). Party survival requires

either that the first two processes be effective enough ($R_1 > 1$ and $R_2 > 1$) for a small grassroots effort to take hold, or else that an initial membership core be large enough, and the second and third processes effective enough ($R_3 > 1$), that direct recruitment by party activists can sustain its membership.

4. **A Case Study: The Green Party.**

4.1. **Methods.** As an application of the simplified one-track model analyzed in Section 3, we used Green Party registration and voting records for six states (CA, ME, MD, NY, OR, PA) and the District of Columbia (DC) to study the growth of the Green Party during the past decade or so.

We began by establishing basic demographic information for the target (study) population. A report from the Pew Research Center [29] provided annual statistics on percentages of Americans identifying themselves as confirmed Democrats or Republicans, leaning Democrat or Republican, or confirmed or leaning toward third parties, for the period 1987–2007. We took an average of these percentages during the period 2000–2007, during which time approximately 25% of the voting population identified itself as leaning Democrat or Republican, and an additional about 11% as confirmed or leaning toward third parties (including the Green Party). We estimate our target population—those capable of being influenced to vote Green— as all of the former group, and about half of the latter group (since the latter group also includes confirmed Greens). Thus we estimate that the target population consists of about 25%+6%=31% of the total voting population of each state. For each state, we averaged the voting population size over the time period of interest (which varied slightly from state to state, as detailed below) from voting records. We then normalized the voting and registration records as proportions of the target population in each case.

To determine the replacement (or mortality) rate $\mu$, we used the average 2003 life expectancy at birth in the U.S. of 77.5 years given in [32], and the minimum voting age of 18 years, to derive an average voting lifetime of 59.5 years, for an estimate of $\mu = 1/59.5\,yr \approx 4.58 \times 10^{-5}/day$. (A United Nations report gives an average U.S. life expectancy at birth of 78.3 years for the period 2000–2005 [30], which yields a comparable estimate of $\mu = 4.77 \times 10^{-5}/day$.)

The data used for this case study came from official voter registration records [4, 9, 17, 20, 21, 25, 27] and election results [3, 8, 16, 19, 22, 26, 28] from each state. Voter registration records showing Green Party registration totals, given in some states as often as monthly, were fit to the size of the member class $M$ over time. Since the dates for which Green Party registration data were given varied from state to state, the initial and final times did as well, but covered approximately the decade 1999–2008. Since voting data were given less often (in general in November of even-numbered years) these data were used, where available, as initial conditions $V(0) + M(0)$. In other cases $V(0)$ was estimated along with other model parameters as discussed below. Table 5 gives the time periods modeled as well as the initial conditions used. The size of each class is given as a percentage of the total target population (the size of which is also given in the table). The rescaled Green Party registration data is also shown in Figure 5. As can be seen in the graph, some states saw a noticeable change in the Green Party's trajectory following the November 2004 election, and so the data for these states (DC, CA, OR, NY) was broken into two subseries (three in the case of DC, which also underwent a visible change following the 2006 election), with different parameter estimates for each subseries.

| State | Initial time | Final time | Initial Conditions $(S(0), V(0), M(0))$ | $\beta - \phi$ | $\alpha\beta$ | $\epsilon$ | $\gamma$ |
|---|---|---|---|---|---|---|---|
| ME | Jun 1998 | Nov 2006 | (91.12, 8.58, 0.31) | 5.27E–4 | 0.5022 | 1.00E–5 | 1.34E–4 |
| DC | Jun 2003 | Sep 2004 | (87.54, 8.61, 3.85) | 0.0181 | 1.9911 | 1.02 | 6.72E–4 |
|  | Jul 2005 | Aug 2006 | (79.52, 16.43*, 4.05) | 1.06E–3 | 0.200 | 0.468 | 0.0115 |
|  | Jan 2007 | Nov 2008 | (89.04, 6.63*, 4.33) | 1.66E–3 | 1.15 | 2.83 | 1.02E–7 |
| CA | Feb 1999 | Feb 2004 | (94.87, 3.09, 2.03) | 4.94E–4 | 0.7188 | 1.69E–5 | 3.73E–4 |
|  | Oct 2004 | May 2008 | (91.87, 4.80*, 3.32) | 0.0105 | 0.2193 | 0.162 | 1.94E–6 |
| OR | Jan 2001 | Oct 2004 | (88.45, 10.26, 1.29) | 1.56E–4 | 1.45 | 2.99E–5 | 1.29E–4 |
|  | Nov 2004 | Mar 2009 | (96.06, 1.70*, 2.23) | 0.0524 | 0.781 | 2.93 | 1.00E–6 |
| NY | Apr 1999 | Mar 2004 | (99.27, 0.73, 0.01) | 0.0119 | 0.352 | 4.00E–3 | 4.33E–4 |
|  | Nov 2004 | Mar 2008 | (98.65, 0.16*, 1.19) | 0.0567 | 0.283 | 3.54 | 1.00E–6 |
| MD | Aug 2000 | Mar 2009 | (98.17, 1.32, 0.01) | 0.0103 | 1.03 | 2.40E–3 | 1.93E–4 |
| PA | Apr 2001 | Nov 2006 | (97.33, 2.55, 0.11) | 1.01E–5 | 0.329 | 1.37E–6 | 9.85E–4 |

TABLE 5. Estimated model parameters, for the seven U.S. states (and district) used in the case study. States are listed in decreasing order of proportional membership $M$ to facilitate comparison with Figure 5. Initial conditions $v(0)$ marked with asterisks *, as well as all model parameters (except $\mu$), were estimated by data fitting as described in the main text. All rates are given in units of 1/days; values below 0.01 are given in scientific notation.
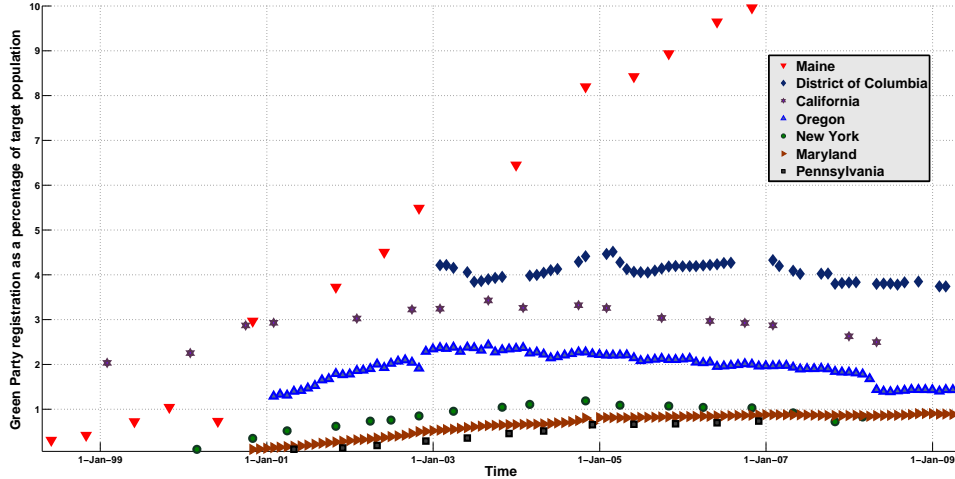


FIGURE 5. Green Party registration data for the seven states (and district) used in the case study, shown as percentages of the target population.

Data fitting used the program Berkeley Madonna 8.3 to obtain a least-squares fit to the Green Party registration data $M(t)$ for the model parameters $\beta - \phi$, $\alpha\beta$, $\epsilon$, and $\gamma$, and, where necessary, initial conditions. ($\beta - \phi$ and $\alpha\beta$ were estimated directly, rather than $\alpha$, $\beta$ and $\phi$ separately, in order to reduce the number of free parameters.) The estimation process was iterated until the estimates were not on target interval boundaries, and this optimization was carried out 10 times for each

| State | Period | $R_1$ | $R_2$ | $R_3$ | Stable equilibria $(v^*, m^*)$ |
|---|---|---|---|---|---|
| ME | 1998–2006 | 9.44 | 2.62 | 43.5 | $E_3 = (34.15,\ 65.84)$ |
| DC | 2003–2004 | 0.18 | – | 72.4 | $E_1 = (0,0)$, $E_3 = (6.81,\ 89.30)$ |
|  | 2005–2006 | 0.0023 | – | 38.9 | $E_1 = (0,0)$, $E_3 = (0.39,\ 98.63)$ |
|  | 2007–2008 | 0.00059 | – | – | $E_1 = (0,\ 0)$ |
| CA | 1999–2004 | 7.88 | 7.11 | 81.3 | $E_3 = (12.28,\ 87.71)$ |
|  | 2004–2008 | 0.065 | – | – | $E_1 = (0,\ 0)$ |
| OR | 2001–2004 | 2.06 | 1.45 | 71.8 | $E_3 = (35.40,\ 64.59)$ |
|  | 2004–2009 | 0.018 | – | – | $E_1 = (0,\ 0)$ |
| NY | 1999–2004 | 2.93 | 6.23 | 53.4 | $E_3 = (10.58,\ 89.27)$ |
|  | 2004–2008 | 0.016 | – | – | $E_1 = (0,\ 0)$ |
| MD | 2000–2009 | 4.20 | 3.21 | 72.1 | $E_3 = (23.76,\ 76.16)$ |
| PA | 2001–2006 | 0.21 | – | 64.8 | $E_1 = (0,0)$, $E_3 = (4.84, 95.14)$ |

TABLE 6. Reproductive numbers and stable equilibria for the one-track model, calculated using parameter estimates from Table 5. Reproductive numbers not given are negative. Equilibria are given as percentages of the target population.

state (with different initial guesses) to obtain the best possible fit. The resulting estimates are also given in Table 5. These estimates were then used to determine the state of the system in each state by calculating reproduction numbers and predicted end states (equilibria); results are shown in Table 6.

4.2. **Results.** As seen in Figure 5, the data indicate a clear period of growth for the Green Party in the selected states during the first five years of the twenty-first century. However, for some states (DC, CA, OR, NY) the data also show a visible decrease beginning at the end of 2004. The data indicate a clear change in the political landscape in these states following the November 2004 election, in which the positive trends the Green Party had been seeing in recent years reversed, and voters left the Green Party, probably for the Democratic Party which saw gains in the 2006 and 2008 elections. The DC Greens rallied in 2005 but experienced this same trend reversal following the 2006 election. In each case, Table 5 suggests that the primary reason for this change was a significant jump in the attention the target population paid to the media and the activities of the two primary political parties, as evidenced by a marked (orders of magnitude) increase in the value of the parameter $\epsilon$ (despite, in most cases, a simultaneous increase in the net peer voter influence $\beta - \phi$).

The other three states in this study (ME, MD, PA), on the other hand, have continued to see healthy sustained growth in the Green Party over the past decade, although the growth in Maine appears nearly meteoric compared to the slow building up in Maryland and Pennsylvania. This difference is reflected in the much higher value of $R_1$ for Maine (cf. Table 6). The parameter estimates for Maine are much lower than for Maryland (cf. Table 5), likely reflecting the lower overall person-to-person contact rate (Maine has no large urban areas, whereas Maryland's population is concentrated in them) as well as perhaps lower sensitivity to others' opinions, but the ratios reflected in the threshold quantities provide fertile ground for the Green Party's growth. Our one-track model predicts (Table 6) a slightly higher equilibrium level of proportional participation in MD than in ME, but it will

take much longer to reach that end state (during which time political influences may change).

Pennsylvania, meanwhile, appears to be in a more unusual situation, as the parameter estimates obtained from the data suggest the presence of a backward bifurcation as discussed in Section 3: Here the tipping-point thresholds $(R_1, R_2)$ do not appear to favor the long-term survival of the Green Party, but an initially large influx of Green voters would allow the party to thrive, even enjoying majority status among the target voting population. The reason for this phenomenon (also seen in the initial periods for DC) can be observed in the parameter estimates, which show a low general sensitivity $\beta - \phi$ to the passive influence of community peers, but a high sensitivity $\alpha\beta$ to grassroots activists (relative to other PA parameters), so that a sufficiently high core of party activists could sustain the party. Despite its initial similarity to Maryland in terms of the proportional data illustrated in Figure 5, the growth of the Pennsylvania Greens is indicated by the model to be part of a transient response—it is just that, in this system, transient responses can last on the order of decades, during which political influences can change significantly.

5. **Analysis of the Two-Track Model.** Having thoroughly examined and interpreted the one-track model, we now perform analysis on the original, heterogeneous two-track model. Since the added complexity of this model precludes a complete qualitative analysis, we shall make use of numerical analysis when necessary to demonstrate behavior analogous to that exhibited by the simpler model.

First, we proportionalize the two-track model in the same way we did for the one-track model to get:

$$\frac{dh}{dt} = p\mu + \epsilon_H v_H + \phi_H(h + \sigma l)v_H - \beta_H(v_H + \sigma v_L + \alpha m)h - \mu h, \tag{21}$$

$$\frac{dl}{dt} = (1-p)\mu + \epsilon_L v_L + \phi_L(l + \sigma h)v_L - \beta_L(\sigma v_H + v_L + \alpha m)l - \mu l, \tag{22}$$

$$\frac{dv_H}{dt} = \beta_H(v_H + \sigma v_L + \alpha m)h - \epsilon_H v_H - \phi_H(h + \sigma l)v_H - \gamma m v_H - \mu v_H, \tag{23}$$

$$\frac{dv_L}{dt} = \beta_L(\sigma v_H + v_L + \alpha m)l - \epsilon_L v_L - \phi_L(l + \sigma h)v_L - \gamma m v_L - \mu v_L, \tag{24}$$

$$\frac{dm}{dt} = \gamma m v_H + \gamma m v_L - \mu m. \tag{25}$$

5.1. **The Party-Free Equilibrium (PFE) and $R_1'$.** The two-track model, like the simpler version, includes a party-free equilibrium. Observing that $v_H^* = v_L^* = m^* = 0$ satisfies $dv_H/dt = dv_L/dt = dm/dt = 0$, we substitute into $dh/dt = dl/dt = 0$ to find the PFE; $E_1$ is $(p, 1-p, 0, 0, 0)$. The stability of $E_1$ is again tied to the first threshold quantity, which we shall denote $R_1'$ (we shall use prime superscripts to denote thresholds for the two-track model) and calculate using the next-generation operator method [5], where $v_H$, $v_L$ and $m$ are considered the infective classes:

$$R_1' = \frac{1}{2}\left[r_{HH} + r_{LL} + \sqrt{(r_{HH} - r_{LL})^2 + 4\,r_{LH}\,r_{HL}}\right], \tag{26}$$

where

$$r_{HH} = p\,\frac{\beta_H - \phi_H}{\mu + \epsilon_H + (1-p)\sigma\phi_H} \quad \text{and} \quad r_{LL} = (1-p)\,\frac{\beta_L - \phi_L}{\mu + \epsilon_L + p\sigma\phi_L},$$

respectively, are the high-affinity and low-affinity analogues of $R_1$ (that is, the influence of $v_H$ on $h$, and of $v_L$ on $l$), and

$$r_{LH} = p \frac{\beta_H \sigma}{\mu + \epsilon_L + p\sigma\phi_L} \quad \text{and} \quad r_{HL} = (1-p) \frac{\beta_L \sigma}{\mu + \epsilon_H + (1-p)\sigma\phi_H}$$

measure the cross-affinity influences (of $v_L$ on $h$ and $v_H$ on $l$). See Appendix B for calculations.

$R_1'$ can be interpreted as the average number of individuals converted into a third-party voter (in either voting class) by an individual in $v_H$ or $v_L$ introduced into a population where no one yet votes for the given party. The average is somewhat complicated, as it involves four different contributing influences, each represented by one of the four $r$'s in (26). Each of these component numbers has the same form as $R_1$ for the one-track model (q.v.), but measures of conversion of high-affinity voters are multiplied by a factor of $p$, the proportion of the population which has a high affinity for the given party, while measures of conversion of low-affinity voters are multiplied by the proportion $1-p$ of low-affinity individuals. In addition, cross-affinity influences are reduced by the factor $\sigma$. Note that in the extreme cases $p = 0$ and $p = 1$ $R_1'$ reduces to $R_1$.

We can further interpret the expression for $R_1'$ in political terms by observing that

$$\max(r_{HH}, r_{LL}) < R_1' < \max(r_{HH}, r_{LL}) + \sqrt{r_{LH}\, r_{HL}} \qquad (27)$$

(again see Appendix B for details). That is, $R_1'$ is at least as great as each of the within-track conversion efficiencies, and exceeds the maximum of the two (presumably $r_{HH}$) by less than the contribution of cross-track influences. This latter contribution is the geometric mean of two terms representing a two-stage process, in which a voter in one track converts an individual of the opposite affinity class into a third-party voter, who then influences an individual in the first track to join the voting class of the original voter, thereby completing the cycle. Because this cross-affinity cycle has two stages, the appropriate measure of its efficiency is a geometric mean of the two individual stages. In the case that there is no cross-affinity influence ($\sigma = 0$), the tracks decouple completely at this stage, and (27) reduces to $R_1' = \max(r_{HH}, r_{LL})$.

The PFE is locally stable when $R_1' < 1$, and unstable when $R_1' > 1$. In other words, each third-party voter introduced into a population that includes high-affinity and low-affinity individuals must influence, on average, more than one person to vote for the third party during his/her voting lifetime, in order for the third-party voter classes to persist, with the average defined by $R_1'$.

### 5.2. The Member-Free Equilibrium (MFE) and $R_2'$.
The two-track model also has a second threshold parameter $R_2'$. Analogous to $R_2$ from the one-track model, we define $R_2'$ as the average number of third-party voters ($V_H$ and/or $V_L$) a member can convert into $M$ if introduced into a population of them. Since $R_2'$ is primarily concerned with the transition from third-party voting to membership, we conveniently regard $M$ as the only infectious class, in order to apply the next-generation operator method to determine this threshold. We then calculate, from (5) (or (25)):

$$\frac{\partial}{\partial M} \left( \frac{dM}{dt} \right) = \gamma(v_H + v_L) - \mu;$$

since this is scalar we seek simply the positive part divided by the term that is subtracted (departing $M$):

$$R_2' = \frac{\gamma}{\mu}\left(v_{H2}^* + v_{L2}^*\right),$$

where $v_{H2}^*$ and $v_{L2}^*$ are the equilibrium values at the MFE. This expression is analogous to that for the one-track model,

$$R_2 = \frac{\gamma}{\mu}v^* = \frac{\gamma}{\mu}\left(1 - \frac{1}{R_1}\right).$$

Solving explicitly for the MFE of the two-track model is complicated, but we can show enough to suggest that, as expected, it is unique and exists only for $R_1' > 1$. Since we have $m^* = 0$, any MFE must be an equilibrium of the subsystem

$$\frac{dv_H}{dt} = \beta_H(v_H + \sigma v_L)(p - v_H) - \phi_H(p - v_H + \sigma(1 - p - v_L))v_H - (\mu + \epsilon_H)v_H,$$
(28)

$$\frac{dv_L}{dt} = \beta_L(\sigma v_H + v_L)(1 - p - v_L) - \phi_L(1 - p - v_L + \sigma(p - v_H))v_L - (\mu + \epsilon_L)v_L,$$
(29)

since we can now rewrite $h = p - v_H$ and $l = 1 - p - v_L$. The two resulting equilibrium conditions can be simplified to a single equation of degree 4, which admits up to 4 solutions. One of these is the PFE, which can be factored out to leave a cubic equation for the MFE. It can be shown that the constant term in this cubic equation is zero precisely when $R_1' = 1$, so that the number of positive solutions changes by one when $R_1'$ crosses 1.

In the special case that $\sigma = 0$ (no cross-affinity influence), the system (28)–(29) decouples, yielding equilibria $E_1(0,0)$, $E_{2a}(\tilde{v}_H^*, 0)$, $E_{2b}(0, \tilde{v}_L^*)$, and $E_{2c}(\tilde{v}_H^*, \tilde{v}_L^*)$, where

$$\tilde{v}_H^* = p - \frac{\mu + \epsilon_H}{\beta_H - \phi_H} \quad \text{and} \quad \tilde{v}_L^* = (1 - p) - \frac{\mu + \epsilon_L}{\beta_L - \phi_L}$$

are meaningful only if positive, i.e., if $r_{HH} > 1$ and $r_{LL} > 1$, respectively. A straightforward calculation of the Jacobian matrix shows that, within this subsystem,

- if $R_1' < 1$ ($r_{HH} < 1$ and $r_{LL} < 1$), the only equilibrium is $E_1$, which is locally asymptotically stable (LAS);
- if $r_{HH} > 1$ and $r_{LL} < 1$, $E_1$ is unstable but $E_{2a}$ is LAS;
- if $r_{HH} < 1$ and $r_{LL} > 1$, $E_1$ is unstable but $E_{2b}$ is LAS;
- if $r_{HH} > 1$ and $r_{LL} > 1$, $E_1$, $E_{2a}$ and $E_{2b}$ are all unstable but $E_{2c}$ is LAS.

Regardless of the value of $\sigma$, it is straightforward to show that all solutions of (28)–(29) which begin within $[0, p] \times [0, 1 - p]$ remain within those bounds, by observing that $dv_H/dt < 0$ when $v_H = p$ and $0 \le v_L \le 1 - p$, and that $dv_L/dt < 0$ when $v_L = 1 - p$ and $0 \le v_H \le p$. One can also exclude limit cycles from solutions of (28)–(29) under the assumptions that $\beta_H > \phi_H$ and $\beta_L > \phi_L$, via the usual application of Dulac's Criterion:

$$\frac{\partial}{\partial v_H}\left(\frac{1}{v_H v_L}\frac{dv_H}{dt}\right) < 0, \quad \frac{\partial}{\partial v_L}\left(\frac{1}{v_H v_L}\frac{dv_L}{dt}\right) < 0.$$

We can therefore apply the Poincaré-Bendixson Theorem to conclude that the equilibria of this system with $\sigma = 0$ identified above as locally stable, are in fact globally stable.
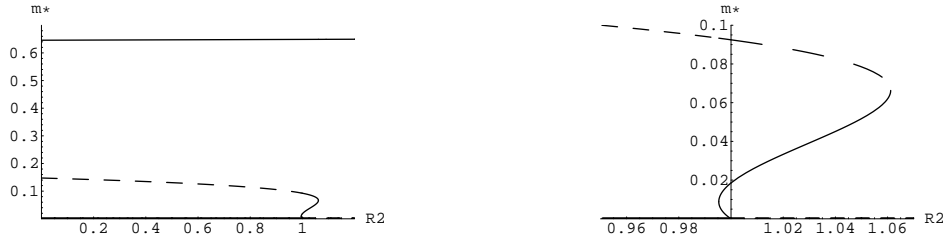
FIGURE 6. A bifurcation diagram for the two-track model illustrating a backward bifurcation at $R_2' = 1$ (see closer view at right), multiple saddle-node bifurcations, and survival equilibria extending below $R_1' = 1$ ($R_2' = 0$). Parameter values used are $p = 0.1$, $\beta_L = 0.2$, $\epsilon_H = 0.25$, $\epsilon_L = 0.5$, $\phi_H = \phi_L = 0.1875$, $\alpha = 1.25$, $\sigma = 0.8$, $\gamma = 0.6$, $\mu = 0.05$; $\beta_H$ varies. All rates are in years$^{-1}$ ($p$, $\alpha$, $\sigma$ are dimensionless).

We can also differentiate the equilibrium conditions for (28)–(29) implicitly by $\sigma$ to see what happens as $\sigma$ increases from zero: for instance,

$$\left.\frac{\partial v_L}{\partial \sigma}\right|_{\sigma=0} = \frac{-\beta_L v_H^*(1 - p - v_L^*) + \phi_L v_L^*(p - v_H^*)}{(\beta_L - \phi_L)(1 - p - 2v_L^*) - (\mu + \epsilon_L)},$$

so

$$\left.\frac{\partial v_L}{\partial \sigma}\right|_{E_{2a},\sigma=0} = \frac{-\beta_L \tilde{v}_H^*(1 - p)}{(\beta_L - \phi_L)(1 - p) - (\mu + \epsilon_L)}.$$

Thus when $r_{HH} > 1$ and $r_{LL} > 1$, the numerator is negative and the denominator is positive, so that $E_{2a}$ exits the state space as $\sigma$ increases from zero. A similar calculation holds for $E_{2b}$.

Since (28)–(29) is a subsystem of (21)–(25) (in which $m(t) \equiv 0$), stability in the subsystem does not imply stability in the full system, but instability in the subsystem does imply instability in the full system. Thus the full system (21)–(25) has at most one stable MFE when $\sigma = 0$ (and, by continuity using the above result of implicit differentiation, for $\sigma$ sufficiently small), that stability depending upon the additional dimension ($m$) not present in (28)–(29), as measured by $R_2'$. While $R_2'$ is not expressed explicitly, given the implicitness of $v_{H2}^*$ and $v_{L2}^*$, the observed uniqueness of the MFE allows us to draw conclusions about the two-track model numerically.

5.3. **Survival Equilibria.** Although the equilibrium conditions for (21)–(25) are too complicated to solve outright (apart from the fact that $v_H^* + v_L^* = \mu/\gamma$ for any survival equilibrium, from (25)), we can verify numerically not only their existence as expected when $R_2' > 1$, but also the existence, under certain conditions, of a backward bifurcation at $R_2' = 1$ just as observed for the one-track model.

Figure 6 shows a situation analogous to that depicted in Figure 4 for the one-track model, in which survival equilibria may exist even below the PFE/MFE threshold ($R_2 = 0$ or $R_2' = 0$); in Figure 7 the critical (minimum) value of $R_2'$ is between 0 and 1. Both figures, however, also demonstrate the existence of multiple locally stable survival equilibria near $R_2' = 1$ (see the close-up in Figure 6), meaning that for some parameter values there are two different levels at which the party may stabilize, in addition to the stable (for $R_2' < 1$) extinction equilibria PFE/MFE. In these
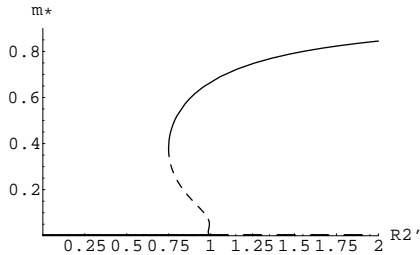
FIGURE 7. A bifurcation diagram for the two-track model illustrating a backward bifurcation at $R_2' = 1$. Parameter values used are $p = 0.1$, $\beta_H = 10$, $\beta_L = 0.2$, $\epsilon_H = 0.25$, $\epsilon_L = 0.5$, $\phi_H = \phi_L = 0.1875$, $\alpha = 1.25$, $\sigma = 0.8$, $\mu = 0.05$; $\gamma$ varies. All rates are in years$^{-1}$ ($p$, $\alpha$, $\sigma$ are dimensionless).

situations, the initial number of party members plays a huge role in determining whether the party surges to major growth, languishes, or dies out entirely.

Although the parameter values used to create these figures are idealized for illustration purposes, the value $p = 0.1$ reflects an estimated 10% of the voting population having high affinity for the Green Party's agenda, and other values reflect the distinctions in peer-driven behavior described by the model for the two affinity classes. In addition, similar curves can be obtained using a wide range of values for model parameters. Unfortunately, the complexity of the two-track model prevents an explicit calculation of a quantity analogous to $R_3$ which measures the ability of the member class to recruit "susceptible" voters from both affinity classes to become third-party voters. The same interpretation applies, however: it is the work of party members in recruiting voters that enables a party to persist when third-party voters' influence is too weak.

6. **Conclusion.** The models described in this study investigate in mathematical terms the consequences of our assumptions about the factors driving the growth and persistence of a third political party arising through grassroots efforts. In particular, we assume a hierarchical structure in which party members (activists) play a different, more extensive role in party survival than party voters. We also assume that the dominant influences are primary contacts among third-party voters and members and the general public, manifested in our models as nonlinear terms involving the sizes of the two groups making contact. These nonlinearities govern the behavior of the models—that is, the fate of the party under study—through threshold quantities that describe the system's tipping points. Our results should be taken as implications of the assumptions that such primary contacts, and not other factors (apart from affinity as defined for our two-track model), drive individuals' decisions whether to support third political parties at any level.

The primary result of our analysis is that our models identify, and provide a way to measure, the three factors that determine the party's survival. Each of these factors is a reproductive number that describes the ability of a given class in the party structure to recruit others. $R_1$ and $R_1'$ give the average number of unaffiliated voters recruited per third-party voter, for the simplified and general model, respectively. This first threshold quantity measures the voting class's ability to replace or sustain itself. $R_2$ and $R_2'$, meanwhile, describes the efficiency with

which third-party members convince third-party voters to become members. This average number of new members recruited per existing member presupposes the success of the first stage of recruitment: existing third-party voters recruiting new ones; that is, $R_1 > 1$, or else $R_2 < 0$, which is meaningless except to indicate the failure of a recruitment structure in which party members play no role in recruiting new voters. Finally, $R_3$ quantifies the ability of party members to recruit new voters directly from the unaffiliated (with this party) public. The growth of the Green Party in states like Maine and Maryland, and its recent decline in states like California and Oregon, illustrate the effect of these tipping-point thresholds (cf. [10]).

The model's prediction of survival states in scenarios where the primary, hierarchical recruitment structure is not strong enough ($R_2 < 1$, and even $R_1 < 1$) to allow a party to arise—that is, political trends are not favorable and the resulting influence of peer pressure to vote for and support a given party is weak—explains the persistence of established third parties during periods of adverse conditions, when political winds blow against them. In particular, when party activists (members) are sufficiently capable of finding and recruiting new voters directly (as measured by $R_3 > 1$), a large enough core of committed party members can ensure the party's survival. In cases where the political environment was favorable for a time ($R_2 > 1$), this minimum core size is typically reached quickly. This reaching across traditional hierarchical structures (rather than party members interacting primarily with those who already vote for the party) provides a robustness to the phenomenon that manifests mathematically in the backward bifurcations illustrated in earlier sections. Backward bifurcations also underscore the importance of initial conditions (having enough initial party members) in enabling that robustness. This scenario is typified by the case study of the Green Party in Pennsylvania.

Our general model classifies the general voting population by affinity to the ideas and goals of a given third party. The form of the expression for $R_1'$ illustrates how a party's ability to take hold in even a small subset of the population (the high-affinity track) affects the party's survival in the population as a whole: since $R_1'$ is greater than either of the within-track voting replacement numbers (analogous to $R_1$), a successful enough recruitment within the high-affinity track can maintain the party. Furthermore, the stratification into two tracks creates the potential, as illustrated in Figures 6 and 7, for multiple stable survival states, which facilitate party growth since the additional states require fewer initial members than the original one. Our models could easily be extended to a stratification with intermediate levels of affinity, each of which would make its own differential ($R_1$-like) contribution to determining the overall recruitment potential of a given third party (analogous to $R_1'$), in addition to increasing the diversity of possible survival states.

Even though we used available data to estimate parameters, we cannot claim to have measured the strength of the various influences directly. Our models provide qualitative measures for the efficiency of parties' recruitment strategies, identifying which factors interact, and how. The accuracy of these measures hinges, of course, on the underlying assumptions articulated in Section 2.1. Since it is often difficult in practice to quantify the strength of another's opinions and arguments in influencing one's opinion, our models can also be interpreted as an illustration of how individual interactions within a population combine to exert a single collective influence observable only at the population level. Finally, it should be noted that the

deterministic nature of our models ignores the stochastic aspect of individual interactions (such as an exceptionally charismatic individual): individual variability is a critical factor when considering very small groups. Here we have described political behavior in terms of population averages, but the first stages of any movement are entirely dependent on the particular personalities involved. Therefore our models should be considered as picking up at the point where a grassroots movement has become sufficiently organized to become a political force.

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## Appendix A. Equilibrium Analysis for the One-Track Model.

### A.1. Proof of Proposition 1: Existence of $E_3$ and $E_4$. We begin with the survival equilibrium condition $f(m^*) = m^{*2} + BM^* + C = 0$, where (from (15))

$$B = \left(\frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - \left(1 - \frac{\mu}{\gamma}\right) + \frac{\mu}{\alpha\beta}, \quad C = -\frac{\mu}{\gamma}\left[\frac{\beta - \phi}{\alpha\beta}\left(1 - \frac{\mu}{\gamma}\right) - \frac{\mu + \epsilon}{\alpha\beta}\right].$$

We rewrite these expressions by defining the following terms:

$$q = \frac{\beta - \phi}{\alpha\beta}, \quad r = \frac{\mu}{\mu + \epsilon}, \quad x = \frac{\alpha\beta}{\mu + \epsilon}, \quad y = \frac{\gamma}{\mu}.$$

By assumption, $\beta > \phi$, $\gamma > \mu$, and $\alpha > 1$, so that $0 < q < 1$, $0 < r < 1$, and $y > 1$. In these terms, $R_1 = qx$, $R_2 = y\left(1 - \frac{1}{qx}\right)$, and

$$B = \frac{1 + q}{y} - \left(1 - \frac{r}{x}\right), \quad C = -\frac{q}{y}\left(1 - \frac{1}{y} - \frac{1}{qx}\right).$$

In order to be meaningful, solutions must fall within the interval $(0, 1 - v^*]$ (since $m^* > 1 - v^*$ will make $s^* = 1 - v^* - m^* < 0$). We calculate

$$f(1 - v^*) = f\left(1 - \frac{1}{y}\right) = \frac{r}{x} + \frac{1 - r}{xy} > 0$$

and

$$f'(1 - v^*) = 2(1 - v^*) + B = 1 - \frac{1 - q}{y} + \frac{r}{x} > 0 \quad (\text{since } 1 - q < 1 < y).$$

These two inequalities imply that any roots of $f$ lie to the left of $1 - v^*$. Next note that $f(0) = C = -\frac{q}{y^2}(R_2 - 1)$. Then when $R_2 > 1$, $f(0) < 0$, so there is exactly one solution $m_+^*$ in $(0, 1 - v^*)$. When $R_2 < 1$, $f(0) > 0$, so the number of solutions in $(0, 1 - v^*)$ is even. In this case, both solutions $m_{\pm}^*$ are in $(0, 1 - v^*)$ precisely when $B^2 - 4C \geq 0$ (the solutions are real) and $B < 0$ (the parabola's vertex $m = -B/2$ lies to the right of 0).

The condition $B < 0$ can be shown equivalent to $x > r$ and $y > \hat{y}_B \equiv (1 + q)\frac{x}{x - r}$. The condition $C > 0$ (i.e., $R_2 < 1$) can be shown equivalent to $qx < 1$ or $y < \hat{y}_C \equiv \frac{qx}{qx - 1}$. The condition $B^2 - 4C \geq 0$ becomes

$$\left(\frac{1 + q}{y} - \frac{x - r}{x}\right)^2 + 4\frac{q}{y}\left(1 - \frac{1}{y} - \frac{1}{qx}\right) \geq 0,$$

and multiplying by $x^2y^2$ we get

$$(x - r)^2 y^2 - 2x\left[(1 - q)(x - r) + 2(1 - qr)\right]y + (1 - q)^2 x^2 \geq 0.$$

Solving the quadratic inequality in $y$, this means $y$ must not be between the two positive roots

$$\hat{y}_{\pm} \equiv \frac{x}{(x - r)^2}\left\{\left[(1 - q)(x - r) + 2(1 - qr)\right] \pm 2\sqrt{(1 - qr)\left[(1 - q)(x - r) + (1 - qr)\right]}\right\}.$$

Thus in order to have two survival equilibria, we must have $x > r$, $\hat{y}_B < y < \hat{y}_C$, and either $y \leq \hat{y}_-$ or $y \geq \hat{y}_+$.

In order to simplify these criteria, we compare the threshold values for $y$. We find that $\hat{y}_- < \hat{y}_B < \hat{y}_+$ is equivalent (after substitution) to

$$\begin{aligned}
-\sqrt{(1 - qr)\left[(1 - q)(x - r) + (1 - qr)\right]} &< qx - 1 \\
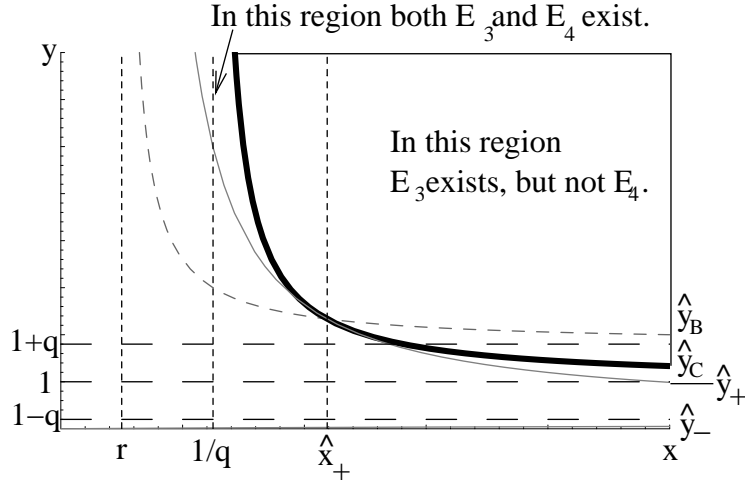&< \sqrt{(1 - qr)\left[(1 - q)(x - r) + (1 - qr)\right]},
\end{aligned} \quad (30)$$

FIGURE 8. Four threshold curves illustrating the survival equilibrium conditions

or simply $|qx - 1| < \sqrt{(1 - qr)\left[(1 - q)(x - r) + (1 - qr)\right]}$. Squaring both sides and simplifying leads to the compound inequality

$$r < x < \hat{x}_+ \equiv \frac{1 + q(1 - r)}{q^2}.$$

The lower bound $r$ corresponds to the vertical asymptote in $x$ shared by $\hat{y}_\pm$ and $\hat{y}_B$. Since $q < 1$, $\frac{1}{q} < \frac{1}{q^2} < \hat{x}_+$, so at the upper bound $qx - 1 > 0$, and it is the second inequality in (30) that is violated. That is, when $x = \hat{x}_+$, $\hat{y}_B = \hat{y}_+$. Thus it is always true that $\hat{y}_- < \hat{y}_B$, so the criterion $y > \hat{y}_B$ allows us to discard the condition $y \leq \hat{y}_-$ in favor of $y \geq \hat{y}_+$.

If we solve the inequality $\hat{y}_B < \hat{y}_C$ for $x$ (when $qx > 1$), we find again the condition $x < \hat{x}_+$, indicating that the graphs of $\hat{y}_B$, $\hat{y}_C$, and $\hat{y}_+$ all cross at $x = \hat{x}_+$. Further similar computation can show that $\hat{y}_+ < \hat{y}_C$ except at $x = \hat{x}_+$, where they are tangent. We thus require $r < x < \hat{x}_+$ and $\hat{y}_+ \leq y < \hat{y}_C$ (the latter inequality only for $qx > 1$). A graph illustrating all four curves is given in Figure 8.

To put the survival equilibrium conditions back in terms of the original parameters, we see that $r < x < \hat{x}_+$ becomes

$$\frac{\mu}{\mu + \epsilon} < \frac{\alpha\beta}{\mu + \epsilon} < \frac{1 + \frac{\beta - \phi}{\mu + \epsilon}\frac{\epsilon}{\mu + \epsilon}}{\left(\frac{\beta - \phi}{\mu + \epsilon}\right)^2};$$

each of the inequalities can be solved for $\alpha\beta/\mu$ to make

$$\frac{\alpha\beta}{\mu} > \max\left(1, \frac{\beta - \phi}{\mu + \epsilon}\frac{\beta - \phi - \epsilon}{\mu}\right).$$

Note that the last expression in the inequality above is the product of $R_1$ and another fraction whose value exceeds 1 precisely when $R_1$ does. Thus if $R_1 < 1$, the condition is simply $\frac{\alpha\beta}{\mu} > 1$, while if $R_1 > 1$ the condition is $\frac{\alpha\beta}{\mu} > R_1(\beta - \phi - \epsilon)/\mu$. It is not so simple to rewrite $y > \hat{y}_+$; however, we can return to the original conditions $B < 0$, $B^2 - 4C \geq 0$, which can be rewritten more simply, either as $R_{3a} > 1$,

$R_{3b} > 1$ as in the statement of Proposition 1, or as:

$$B < 0 \Leftrightarrow \frac{1}{\alpha\beta} + \left(1 + \frac{\beta - \phi}{\alpha\beta}\right)\frac{1}{\gamma} < \frac{1}{\mu}, \tag{31}$$

$$B^2 - 4C \geq 0 \Leftrightarrow \sqrt{\frac{1}{\alpha\beta}} + \sqrt{\left(1 - \frac{\beta - \phi}{\alpha\beta}\right)\frac{1}{\gamma} + h(\epsilon/\gamma)} \leq \sqrt{\frac{1}{\mu}}, \tag{32}$$

where

$$h(\epsilon/\gamma) = 2\sqrt{\frac{1}{\alpha\beta}}\sqrt{\frac{1}{\mu}}\left(\sqrt{1 + \frac{\epsilon}{\gamma}} - 1\right) > 0.$$

In order to rewrite the condition $B^2 - 4C \geq 0$ in the form (32), we substitute $B$ and $C$ from (15) and get:

$$\left[\frac{\mu}{\alpha\beta} + \left(1 + \frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} - 1\right]^2 - 4\frac{\mu}{\gamma}\left[\frac{\mu + \epsilon}{\alpha\beta} - \frac{\beta - \phi}{\alpha\beta}\left(1 - \frac{\mu}{\gamma}\right)\right] \geq 0.$$

Expansion, summing like terms, and completing the square yields

$$\left[\frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1\right]^2 \geq 4\frac{\mu}{\alpha\beta}\left(1 + \frac{\epsilon}{\gamma}\right).$$

Next we take the square root of both sides,

$$\left|\frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1\right| \geq 2\sqrt{\frac{\mu}{\alpha\beta}\left(1 + \frac{\epsilon}{\gamma}\right)}.$$

Since we require $\mu < \gamma$ and $\beta > \phi$, then

$$\frac{\mu}{\alpha\beta} - \left(1 - \frac{\beta - \phi}{\alpha\beta}\right)\frac{\mu}{\gamma} + 1 = \frac{\mu}{\alpha\beta} + \frac{\beta - \phi}{\alpha\beta}\frac{\mu}{\gamma} + \left(1 - \frac{\mu}{\gamma}\right) > 0,$$

so we can drop the absolute value bars. Finally, rearranging, we get

$$\left(1 - \frac{\beta - \phi}{\alpha\beta}\right)\frac{1}{\gamma} \leq \frac{1}{\mu} - 2\sqrt{\frac{1}{\alpha\beta}\frac{1}{\mu}\left(1 + \frac{\epsilon}{\gamma}\right)} + \frac{1}{\alpha\beta}.$$

In the case that $\epsilon = 0$, this inequality can be further simplified by factoring the right-hand side as a perfect square, taking the square root of both sides, and using the fact that (31) implies $1/\alpha\beta < 1/\mu$, to get

$$\sqrt{\frac{1}{\alpha\beta}} + \sqrt{\left(1 - \frac{\beta - \phi}{\alpha\beta}\right)\frac{1}{\gamma}} \leq \sqrt{\frac{1}{\mu}}.$$

We can apply the same technique to obtain (32).

A.2. **Stability Analysis for $E_3$.** From the end of Section 3.3, it remains to show that $\operatorname{tr} J_1(E_3) < 0$ when $R_2 \geq 2$. This condition on the trace is equivalent to

$$m_+^* > L \equiv \frac{(\beta - \phi)\left(1 - 2\frac{\mu}{\gamma}\right) - (\mu + \epsilon)}{(\beta - \phi + \alpha\beta + \gamma)}.$$

In terms of $q$, $r$, $x$ and $y$ as defined in the previous section,

$$L = \frac{q\left(1 - \frac{1}{qx} - \frac{2}{y}\right)}{\left(1 + q + \frac{yr}{x}\right)}, \quad B = \frac{\left(1 + q + \frac{yr}{x}\right)}{y} - 1, \quad C = -\frac{q}{y}\left(1 - \frac{1}{qx} - \frac{1}{y}\right).$$

Now the stability condition is

$$m_+^* = \frac{1}{2}\left[-B + \sqrt{B^2 - 4C}\right] > L$$

$$\sqrt{B^2 - 4C} > B + 2L$$

$$B + 2L < 0, \text{ or } B + 2L > 0 \text{ and } B^2 - 4C > B^2 + 4BL + 4L^2$$

$$B + 2L > 0 \implies -C > BL + L^2$$

$$B + 2L > 0 \implies \frac{q}{y}\left(1 - \frac{1}{qx} - \frac{1}{y}\right) > \frac{q}{y}\left(1 - \frac{1}{qx} - \frac{2}{y}\right) - L + L^2$$

$$B + 2L > 0 \implies 0 > -\frac{q}{y^2} - L + L^2$$

Here we multiply by $-\left(1 + q + \frac{yr}{x}\right)^2/q$, expand, and simplify to get

$$0 < \left(\frac{1+q}{y} + \frac{r}{x}\right)^2 + \left(1 - \frac{1}{qx} - \frac{2}{y}\right)\left(1 + \frac{1+yr}{x} + \frac{2q}{y}\right),$$

which is true since

$$R_2 \geq 2 \Leftrightarrow \left(1 - \frac{1}{qx} - \frac{2}{y}\right) \geq 0.$$

This completes the verification that $E_3$ is LAS when it exists.

**Appendix** B. $R_1'$ **for the Two-Track Model.** We calculate the reproductive number $R_1'$ of the two-track model using the next-generation operator method where $R_1'$ is analogous to $R_1$ of the one-track model. The party-free equilibrium $E_1$ of (21)–(25) is $(p, (1-p), 0, 0, 0)$. Differentiating (23)–(25) with respect to the "infective" variables $v_H$, $v_L$, and $m$, and substituting the PFE values yields the following "mini-Jacobian" matrix:

$$A = \begin{pmatrix} (\beta_H - \phi_H)p - w_H & \beta_H \sigma p & \alpha\beta_H p \\ \beta_L \sigma(1-p) & (\beta_L - \phi_L)(1-p) - w_L & \alpha\beta_L(1-p) \\ 0 & 0 & -\mu \end{pmatrix},$$

where $w_H = \mu + \epsilon_H + \phi_H \sigma(1-p)$ and $w_L = \mu + \epsilon_L + \phi_L \sigma p$. We next rewrite $A = \tilde{M} - \tilde{D}$, where the entries of $\tilde{M}$ are nonnegative and $\tilde{D}$ is a diagonal matrix:

$$\tilde{M} = \begin{pmatrix} (\beta_H - \phi_H)p & \beta_H \sigma p & \alpha\beta_H p \\ \beta_L \sigma(1-p) & (\beta_L - \phi_L)(1-p) & \alpha\beta_L(1-p) \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} \mu + \epsilon_H + \phi_H \sigma(1-p) & 0 & 0 \\ 0 & \mu + \epsilon_L + \phi_L \sigma p & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

Now $R_1'$ is the dominant (largest) eigenvalue of

$$MD^{-1} = \begin{pmatrix} \frac{(\beta_H - \phi_H)p}{\mu + \epsilon_H + \phi_H \sigma(1-p)} & \frac{\beta_H \sigma p}{\mu + \epsilon_L + \phi_L \sigma p} & \alpha\beta_H p/\mu \\ \frac{\beta_L \sigma(1-p)}{\mu + \epsilon_H + \phi_H \sigma(1-p)} & \frac{(\beta_L - \phi_L)(1-p)}{\mu + \epsilon_L + \phi_L \sigma p} & \alpha\beta_L(1-p)/\mu \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} r_{HH} & r_{LH} & \alpha\beta_H p/\mu \\ r_{HL} & r_{LL} & \alpha\beta_H p/\mu \\ 0 & 0 & 0 \end{pmatrix}.$$

The three eigenvalues are 0 and

$$\frac{1}{2}\left[r_{HH} + r_{LL} \pm \sqrt{(r_{HH} - r_{LL})^2 + 4r_{HL}\,r_{LH}}\right];$$

$R_1'$ takes the positive square root in the latter expression. By inspection we can see that for the extreme cases $p = 0$ and $p = 1$, $R_1'$ simplifies to $R_1$ for the one-track model.

Since the second term inside the radical is positive, we have that

$$R_1' > \frac{1}{2}\left[r_{HH} + r_{LL} + \sqrt{(r_{HH} - r_{LL})^2}\right] = \frac{1}{2}\left[r_{HH} + r_{LL} + |r_{HH} - r_{LL}|\right]$$

$$= \frac{1}{2}\left[2\max(r_{HH}, r_{LL})\right] = \max(r_{HH}, r_{LL}).$$

Since, for positive numbers $a$ and $b$, $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$, we also have that

$$R_1' < \frac{1}{2}\left[r_{HH} + r_{LL} + \sqrt{(r_{HH} - r_{LL})^2} + 2\sqrt{r_{HL}\,r_{LH}}\right]$$

$$= \max(r_{HH}, r_{LL}) + \sqrt{r_{HL}\,r_{LH}}.$$

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